Axiom Schemes in Machover's Propositional Calculus

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1 The Five Axiom Schemes

In the proof system *Propcal*, there are five axiom schemes which, along with a single inference rule *Modus Ponens*, allows for linear proofs to be constructed. Keep in mind that axiom schemes are not axioms themselve — but general forms from which axioms can be instantiated.

Axiom Scheme I	$\alpha \to (\beta \to \alpha)$	(affirming the consequent)
Axiom Scheme II	$[\alpha \to (\beta \to \gamma)] \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$	($\to$ distributivity)
Axiom Scheme III	$[(\alpha \to \beta) \to \alpha] \to \alpha$	(Peirce's Law)
Axiom Scheme IV	$\neg\alpha \to (\alpha \to \beta)$	(denying the antecedent)
Axiom Scheme V	$(\alpha \to \neg\alpha) \to \neg\alpha$	("Reductio" Axiom)

2 Explanations for the Axiom Schemata

Axiom Schema I This captures the law of *affirming the consequent* (i.e., if a proposition is true, then anything implies it). In practice, this axiom scheme is used when one needs to proceed from α to $\beta \to \alpha$ in a linear axiomatic proof system.

Axiom Schema II An axiom scheme often used to distribute the antecedent (and the $\to$) over a consequent which is an implication itself. In the context of a linear proof, it allows us to proceed from $\alpha \to (\beta \to \gamma)$ to $(\alpha \to \beta) \to (\alpha \to \gamma)$, and is instrumental is proving propositions of the form $\alpha \to \alpha$ and the structural rule Deduction Theorem (DT).

Axiom Schema III Otherwise known as *Peirce's Law*, this axiom scheme says that if an implication implies its antecedent, then this antecedent must hold. In a linear proof, it warrants the inference of α from $(\alpha \to \beta) \to \alpha$. It is a scheme adopted to prove propositions of the form $\neg\neg\alpha \to \alpha$, and the associated structural rule Principle of Indirect Proof (PIP).

Axiom Scheme IV This axiom scheme embodies the law of *denying the antecedent* (i.e., if a proposition is false, then it implies anything). It validates the inference of "$\neg\alpha$, hence $\alpha \to \beta$," in a linear proof, and is a key scheme in establishing the structure rule Inconsistency Effect (IE).

Axiom Scheme V An axiom scheme reminiscent of a proof-by-contradiction method called *reductio ad absurdum*, which stipulates that if a propostion leads to some contradiction, then it must be false. In the context of a linear proof, this scheme enables us to proceed from $\alpha \to \neg\alpha$ to $\neg\alpha$, and is instrumental in establishing the structural rule Reductio.