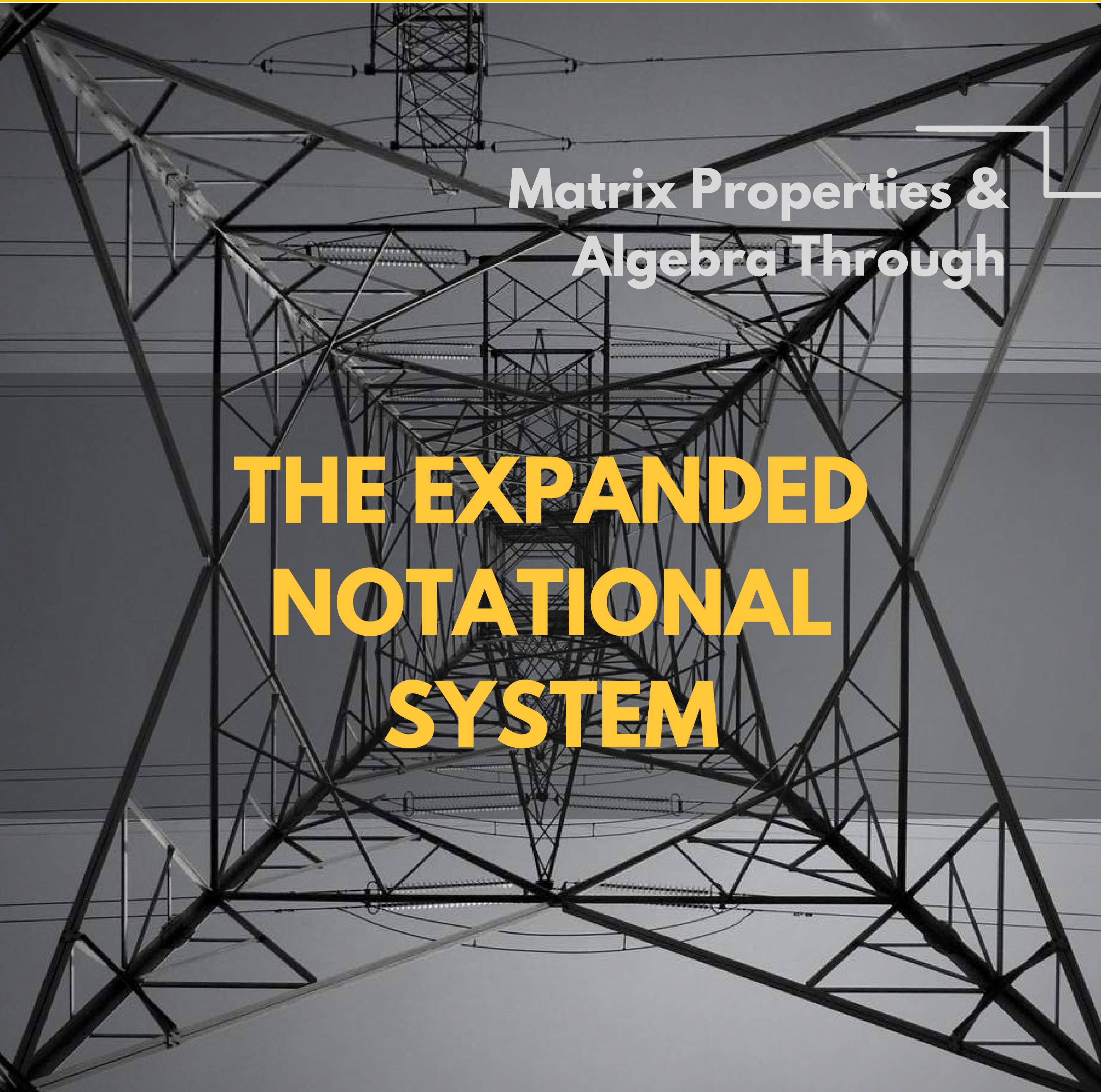

Matrix Properties & Algebra Through

THE EXPANDED NOTATIONAL SYSTEM

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Matrix Properties & Algebra Through The Expanded Notational System

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In this paper, we introduce the **Expanded Notational System** for matrices and its components, and illustrate its theoretical and algebraic advantages over the standard notational system — where expressions are often written *solely* in terms of matrix entries or the matrices themselves .

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1 Preliminaries

In this section, we introduce the various components of the **Expanded Notational System**, along with some exploration around the concept of **matrix equivalence**.

1.1 The Expanded Notational System

As expected, we begin with the standard definition of a matrix:

Definition 1.1 (Matrix)

In linear algebra, we define a **matrix** A as a two-dimensional array of numbers of the form:

$$\begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{pmatrix}$$

with A_{ij} ($1 \leq i \leq m$, $1 \leq j \leq n$) being its individual entries.

Since A has m rows and n columns, it falls into the category of a $m \times n$ matrix. In addition:

- If $m = 1$, then we say that A is a n-entry **row matrix**.
- If $n = 1$, then we say that A is a m-entry **column matrix**.

In some occasions, instead of specifying all the entries of a matrix, we might be more interested in specifying only the **rows** or the **columns** of the matrix. In that case, the following definition can be helpful:

Definition 1.2 (Row/Column Representation of a Matrix)

Given m n-entry row matrices $R_1, \ldots, R_m$ with $R_i = \begin{pmatrix} R_{i1} & \cdots & R_{in} \end{pmatrix}$ for each $i \in \{1, \ldots, m\}$, we define the following expression

$$\begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix}$$

to be the $m \times n$ matrix containing the corresponding entries of the said row matrices. That is:

$$\begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix} \overset{df}{=} \begin{pmatrix} R_{11} & \cdots & R_{1n} \\ \vdots & \ddots & \vdots \\ R_{m1} & \cdots & R_{mn} \end{pmatrix}$$

Similarly, given n m-entry column matrices $C_1, \ldots, C_n$ with $C_j = \begin{pmatrix} C_{1j} \\ \vdots \\ C_{mj} \end{pmatrix}$ for all $j \in \{1, \ldots, n\}$, we define the expression $\begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix}$ as the $m \times n$ matrix containing the corresponding entries of the said column matrices. That is:

$$\begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix} \overset{df}{=} \begin{pmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{m1} & \cdots & C_{mn} \end{pmatrix}$$

Here, notice that **parentheses** are used to specify a matrix by its entries, and **brackets** are used to specify a matrix by its rows or columns. Also note that this definition is unambiguous in the sense that:

- A **vertical representation** always corresponds to a *row representation* of a matrix.
- A **horizontal representation** always corresponds to a *column representation* of a matrix.

With these handy notations, we can now proceed to introduce a notation to “capture" the rows of a matrix:

Definition 1.3 (Matrix Row)

Given a $m \times n$ matrix A, we define $A_{i\square}$ — colloquially the i^{th} **row** of A — as the n-entry row matrix whose entries are precisely those found in A's i^{th} row. That is:

$$A_{i\square} \overset{df}{=} \begin{pmatrix} A_{i1} & \cdots & A_{in} \end{pmatrix} \qquad \text{(where } i \in \{1, \ldots, m\}\text{)}$$

Note that this notation also allows us to rewrite A as its row representation as follows:

$$\begin{bmatrix} A_{1\square} \\ \vdots \\ A_{m\square} \end{bmatrix}$$

As expected, a similar notation can also be applied to capture the columns of a matrix:

Definition 1.4 (Matrix Column)

Given a $m \times n$ matrix A, we define $A_{\square j}$ — colloquially the j^{th} **column** of A — as the m-entry column matrix whose entries are precisely those found under

A's j^{th} column. That is:

$$A_{\square j} \overset{df}{=} \begin{pmatrix} A_{1j} \\ \vdots \\ A_{mj} \end{pmatrix} \qquad\qquad \text{(where } j \in \{1, \ldots, n\}\text{)}$$

Note that this notation also allows us to rewrite A as its column representation as follows:

$$\begin{bmatrix} A_{\square 1} & \cdots & A_{\square n} \end{bmatrix}$$

With these notations set, we now have the ability to not only **construct** a matrix from a series of row matrices or column matrices, but **deconstruct** a matrix into its rows and columns as well.

As a final reminder, note that given a matrix A:

- A_{ij} is always a *number* for admissible i and j.
- $A_{i\square}$ is always a *row matrix* for admissible i.
- $A_{\square j}$ is always a *column matrix* for admissible j.
- $[A_{i\square}] = \begin{pmatrix} A_{i1} & \cdots & A_{in} \end{pmatrix} = A_{i\square}$ for admissible i.
- $[A_{\square j}] = A_{\square j}$ for admissible j.

And with that, we have completed our introduction of the **Expanded Notational System**, where:

- Matrices can be represented in both 1D (**row/column representation**) and 2D (**standard array representation**).
- In addition to matrices and matrix entries, **matrix rows** and **matrix columns** can be referred to by their symbolic counterparts as well.

Remark: Why such notations?

While these notations might look a bit cumbersome at first, it generalizes very well when it comes to **higher-dimensional matrices**. In addition, it helps us refine the algebra on not just the *matrix rows* and *matrix columns*, but on the **row/column representations of matrices** as well.

1.2 Matrix Equivalence

The following definition lays out what it means for two matrices to be equal:

Definition 1.5 Matrix Equivalence

Given a $m \times n$ matrix A and a $m' \times n'$ matrix B, we say that $A = B$ if and only if the following two conditions jointly hold:

1. $m = m'$ and $n = n'$.
2. $A_{ij} = B_{ij}$ for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

Naturally, one can verify matrix equivalence by equating *solely* the **rows** or the **columns** of the matrices as well. The following chain of equivalences is a testament to this fact:

Proposition 1.1 (Matrix Equivalence — Three Symbolic Approaches)

Given two $m \times n$ matrices A and B, the following three claims are equivalent:

(i) $A = B$
(ii) $A_{i\square} = B_{i\square}$ for all $i \in \{1, \ldots, m\}$.
(iii) $A_{\square j} = B_{\square j}$ for all $j \in \{1, \ldots, n\}$.

Proof

(i) $\implies$ (ii)

Since $A = B$, $A_{ij} = B_{ij}$ for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. This means that for all $i \in \{1, \ldots, m\}$:

$$\begin{aligned} A_{i\square} &= \begin{pmatrix} A_{i1} & \cdots & A_{in} \end{pmatrix} \\ &= \begin{pmatrix} B_{i1} & \cdots & B_{in} \end{pmatrix} \\ &= B_{i\square} \end{aligned}$$

(ii) $\implies$ (i)

Since (ii) holds, given any $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, m\}$ we must have that:

$$\begin{aligned} A_{i\square} &= B_{i\square} && \iff \\ \begin{pmatrix} A_{i1} & \cdots & A_{in} \end{pmatrix} &= \begin{pmatrix} B_{i1} & \cdots & B_{in} \end{pmatrix} && \implies \\ A_{ij} &= B_{ij} \end{aligned}$$

The proofs for (i) $\implies$ (iii) and (iii) $\implies$ (i) can be carried out in a similar manner, yielding the chain of equivalences as desired. ∎

The following proposition — which allows us to perform algebra on row/column representations of matrices — represents a slightly different rendition of the just-established Proposition 1.1:

Proposition 1.2 (Matrix Equivalence — Visual Approaches)

Given m n-entry row matrices $R_1, \dots, R_m$ and another set of m n-entry row matrices $R'_1, \dots, R'_m$, we have that:

$$\begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix} = \begin{bmatrix} R'_1 \\ \vdots \\ R'_m \end{bmatrix} \qquad \iff \qquad R_i = R'_i \text{ for all } i \in \{1, \dots, m\}$$

Similarly, given n m-entry column matrices $C_1, \dots, C_n$ and another set of n m-entry column matrices $C'_1, \dots, C'_n$, we also have that:

$$\begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix} = \begin{bmatrix} C'_1 & \cdots & C'_n \end{bmatrix} \qquad \iff \qquad C_j = C'_j \text{ for all } j \in \{1, \dots, n\}$$

Proof Immediate from Proposition 1.1. ■

In practice, the **symbolic approaches** for establishing matrix equivalence tend to be more concise, while the **visual approaches** more intuitive, and while both approaches provide two additional techniques for proving matrix equivalence, it remains that the standard technique can still be very helpful when applied at the right place.

2 Operations

In what follows, we'll look at some of the most common matrix operations and how they are handled under the Expanded Notational System. These operations include **scalar multiplication**, **addition**, **multiplication** and **transposition**.

2.1 Scalar Multiplication

As usual, we begin with the standard definition of scalar multiplication:

Definition 2.1 (Scalar Multiplication)

Given a $m \times n$ matrix and a number k, we define kA as the $m \times n$ matrix resulted from multiplying each entry of A by k. That is:

$$(kA)_{ij} \overset{\text{df}}{=} kA_{ij} \qquad \text{(for all } i \in \{1, \ldots, m\} \text{ and } j \in \{1, \ldots, n\})$$

Algebraically, this means that when it comes to scalar products, we can elect to push the **matrix indices** in or out. In fact, the following proposition shows that more is true:

Proposition 2.1 (Scalar Multiplication on Row/Column Representations)

Given m n-entry row matrices $R_1, \ldots, R_m$ and a number k, we have that:

$$k \begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix} = \begin{bmatrix} kR_1 \\ \vdots \\ kR_m \end{bmatrix}$$

Similarly, given n m-entry column matrices $C_1, \ldots, C_n$ and a number k, we also have that:

$$k \begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix} = \begin{bmatrix} kC_1 & \cdots & kC_n \end{bmatrix}$$

Proof To begin, notice that for both claims, the expressions on either sides of the equation are both $m \times n$ matrices, For the first claim, we can proceed by equating the **corresponding rows** of the two matrices.

More specifically, given an $i \in \{1, \ldots, m\}$ with $R_i = \begin{pmatrix} R_{i1} & \ldots & R_{in} \end{pmatrix}$, we can see that:

$$\begin{aligned} \left(k \begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix} \right)_{i\square} &= \begin{pmatrix} kR_{i1} & \ldots & kR_{in} \end{pmatrix} \\ &= k \begin{pmatrix} R_{i1} & \cdots & R_{in} \end{pmatrix} \\ &= kR_i \\ &= \begin{bmatrix} kR_1 \\ \vdots \\ kR_m \end{bmatrix}_{i\square} \end{aligned}$$

And since i is arbitrary, the proof of the first claim is complete. Note that the second claim can be proved similarly by equating the **corresponding columns** of the matrices.

■

In particular, we have that:

Proposition 2.2 (Scalar Multiplication on Matrix Rows and Matrix Columns)

Given a $m \times n$ matrix A, the following two claims hold:

(i) $(kA)_{i\square} = kA_{i\square}$ for all $i \in \{1, \ldots, m\}$.

(ii) $(kA)_{\square j} = kA_{\square j}$ for all $j \in \{1, \ldots, n\}$.

Proof For (i), notice that given any $i \in \{1, \ldots, m\}$, we have that:

$$\begin{aligned}(kA)_{i\square} &= \left(k \begin{bmatrix} A_{1\square} \\ \vdots \\ A_{m\square} \end{bmatrix} \right)_{i\square} \\ &= \begin{bmatrix} kA_{1\square} \\ \vdots \\ kA_{m\square} \end{bmatrix}_{i\square} \\ &= kA_{i\square}\end{aligned}$$

By breaking A down into its **column representation**, we can also prove (ii) in a very similar manner. ■

In other words, not only can we push the indices in and out when it comes to matrix entries, but we can also do so with any **matrix column** or **matrix row** as well.

Exercise Given a $m \times n$ matrix A and two numbers k_1 and k_2, prove that $k_1(k_2A) = (k_1k_2)A$ (also known as **scalar associativity**) using the following techniques:

(i) By equating the corresponding **matrix entries** (symbolic approach).

(ii) By equating the corresponding **matrix rows** (both symbolic and visual approaches).

(iii) By equating the corresponding **matrix columns** (both symbolic and visual approaches).

Among the five techniques, which one did you find to be the most elegant in this case? And why?

2.2 Addition

As usual, we begin with the standard definition of matrix addition:

Definition 2.2 (Addition)

Given two $m \times n$ matrices, we define $A + B$ as the $m \times n$ matrix resulted from adding the corresponding entries of A and B. That is:

$$(A+B)_{ij} \overset{df}{=} A_{ij} + B_{ij} \qquad \text{(for all } i \in \{1, \ldots, m\} \text{ and } j \in \{1, \ldots, n\}\text{)}$$

Algebraically, this means that when it comes to the sum of matrices, the matrix indices can be **distribute over** or **factor out** of the two matrices. In fact, the following proposition shows that more is true:

Proposition 2.3 (Addition on Row/Column Representations)

Given m n-entry row matrices $R_1, \ldots, R_m$ and another set of m n-entry row matrices $R'_1, \ldots, R'_m$, we have that:

$$\begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix} + \begin{bmatrix} R'_1 \\ \vdots \\ R'_m \end{bmatrix} = \begin{bmatrix} R_1 + R'_1 \\ \vdots \\ R_m + R'_m \end{bmatrix}$$

Similarly, given n m-entry column matrices $C_1, \ldots, C_n$ and another set of n m-entry column matrices $C'_1, \ldots, C'_n$, we also have that:

$$\begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix} + \begin{bmatrix} C'_1 & \cdots & C'_n \end{bmatrix} = \begin{bmatrix} C_1 + C'_1 & \cdots & C_n + C'_n \end{bmatrix}$$

Proof To begin, notice that for both claims, the expressions on either sides of the equation are both $m \times n$ matrices. For the first claim, we can proceed by equating the **corresponding rows** of the two matrices.

More specifically, given an $i \in \{1, \ldots, m\}$ with $R_i = \begin{pmatrix} R_{i1} & \cdots & R_{in} \end{pmatrix}$ and $R'_i = \begin{pmatrix} R'_{i1} & \cdots & R'_{in} \end{pmatrix}$, we can see that:

$$\left(\begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix} + \begin{bmatrix} R'_1 \\ \vdots \\ R'_m \end{bmatrix}\right)_{i\square} = \begin{pmatrix} R_{i1} + R'_{i1} & \cdots & R_{in} + R'_{in} \end{pmatrix}$$
$$= R_i + R'_i$$
$$= \begin{bmatrix} R_1 + R'_1 \\ \vdots \\ R_m + R'_m \end{bmatrix}_{i\square}$$

And since i is arbitrary, the proof of the first claim is complete. Note that the second claim can be proved similarly by equating the **corresponding columns** of the matrices.

■

In particular, we also have that:

Proposition 2.4 (Addition on Matrix Rows and Matrix Columns)

Given two $m \times n$ matrices A and B, the following two claims hold:

(i) $(A+B)_{i\square} = A_{i\square} + B_{i\square}$ for all $i \in \{1, \ldots, m\}$.
(ii) $(A+B)_{\square j} = A_{\square j} + B_{\square j}$ for all $j \in \{1, \ldots, n\}$.

Proof For (i), notice that given any $i \in \{1, \ldots, m\}$, we have that:

$$(A+B)_{i\square} = \left(\begin{bmatrix} A_{1\square} \\ \vdots \\ A_{m\square} \end{bmatrix} + \begin{bmatrix} B_{1\square} \\ \vdots \\ B_{m\square} \end{bmatrix}\right)_{i\square}$$
$$= \begin{bmatrix} A_{1\square} + B_{1\square} \\ \vdots \\ A_{m\square} + B_{m\square} \end{bmatrix}_{i\square}$$
$$= A_{i\square} + B_{i\square}$$

By breaking A and B down into their **column representations**, we can also prove (ii) in a very similar manner.

■

In other words, not only can we **distribute** and **factor out** the indices associated with matrix entries, but we can also do so with the indices associated with **matrix columns** and **matrix rows** as well.

Exercise Using the three symbolic and the two visual techniques for establishing matrix equivalence, prove that given three $m \times n$ matrices A, B and C and three numbers k, k_1 and k_2, the following properties on matrices hold:

(i) $A + B = B + A$ (**Commutativity on $+$**)

(ii) $(A + B) + C = A + (B + C)$ (**Associativity on $+$**)

(iii) $k(A + B) = kA + kB$ (**Scalar Distributivity — Two Matrices**)

(iv) $(k_1 + k_2)A = k_1A + k_2A$ (**Scalar Distributivity — Two Constants**)

Among the five techniques, which would you consider to be the most elegant? Why?

2.3 Transposition

At this point, the matrix operations covered thus far are all **binary** in nature, in that they require *two* inputs in order to produce the intended output. In what follows, we'll introduce our very first **unary** matrix operation — an operation whose output only requires *one* input:

Definition 2.3 (Transposition)

Given a $m \times n$ matrix A, we define A^T — the **transpose** of A — as the $n \times m$ matrix resulted from interchanging the row and column numbers of A's entries. That is:

$$(A^T)_{ij} = A_{ji} \qquad \text{(for all } i \in \{1, \ldots, n\} \text{ and } j \in \{1, \ldots, m\}\text{)}$$

While transposition might seem a bit counter-intuitive at first, the following proposition shows that it is in actuality grounded in some simple geometric intuition one can readily make use of:

Proposition 2.5 (Transposition on Row/Column Representations)

Given m n-entry row matrices $R_1, \ldots, R_m$, we have that:

$$\begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix}^T = \begin{bmatrix} R_1{}^T & \cdots & R_m{}^T \end{bmatrix}$$

Similarly, given n m-entry column matrices $C_1, \ldots, C_n$, we also have that:

$$\begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix}^T = \begin{bmatrix} {C_1}^T \\ \vdots \\ {C_n}^T \end{bmatrix}$$

Proof To begin, notice that for both claims, the expressions on either sides of the equation are both $n \times m$ matrices. For the first claim, we can proceed by equating the **corresponding entries** of the two matrices.

More specifically, given an $i \in \{1, \ldots, n\}$ and a $j \in \{1, \ldots, m\}$ with $R_j = \begin{pmatrix} R_{j1} & \cdots & R_{jn} \end{pmatrix}$, exploring the left side of the equation yields that:

$$\begin{aligned} \left(\begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix}^T \right)_{ij} &= \begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix}_{ji} \\ &= R_{ji} \end{aligned}$$

On the other hand, since ${R_j}^T = \begin{pmatrix} R_{j1} \\ \vdots \\ R_{jn} \end{pmatrix}$, exploring the right side of the equation also yields that:

$$\begin{bmatrix} {R_1}^T & \cdots & {R_m}^T \end{bmatrix}_{ij} = R_{ji}$$

And since i and j are arbitrary, the proof of the first claim is complete. Note that the second claim can be proved similarly by equating the **corresponding entries** of the two matrices as well.

■

In other words, transposition turns the rows of a matrix into its columns, and the columns of a matrix into its rows. Here is a proposition that articulates the same idea — albeit from a slightly different perspective:

Proposition 2.6 (Transposition on Matrix Rows and Matrix Columns)

Given a $m \times n$ matrix A, the following two claims hold:

(i) $(A^T)_{i\square} = (A_{\square i})^T$ for all $i \in \{1, \ldots, n\}$.
(ii) $(A^T)_{\square j} = (A_{j\square})^T$ for all $j \in \{1, \ldots, m\}$.

Proof For (i), notice that given any $i \in \{1, \ldots, n\}$, we have that:

$$\begin{aligned}(A^T)_{i\square} &= \left(\begin{bmatrix} A_{\square 1} & \cdots & A_{\square n}\end{bmatrix}^T\right)_{i\square} \\ &= \begin{bmatrix} (A_{\square 1})^T \\ \vdots \\ (A_{\square n})^T \end{bmatrix}_{i\square} \\ &= (A_{\square i})^T\end{aligned}$$

By breaking A down into its **row representation**, we can also prove (ii) in a very similar manner. ■

As we have seen, the algebra on transposition is a bit more subtle, and involves both **swapping** and **distributing/factoring**. On the top of that, the transposition operator can be also combined with other previously-mentioned operators — yielding a new series of interesting matrix properties (see the exercise below).

Exercise Prove — using an appropriate technique in each case — that given two $m \times n$ matrices A and B and a number k, the following properties on matrices hold:

(i) $(A + B)^T = A^T + B^T$
(ii) $(kA)^T = k(A^T)$
(iii) $(A^T)^T = A$

2.4 Multiplication

Unlike other operations, matrix multiplication is intricately defined and plays a foundational role in many subtopics of linear algebra. As such, we'll begin its treatment by first introducing the basic definitions and properties — before embarking on other key properties where **multiple operators** are involved.

2.4.1 Basic Definitions and Properties

Unlike scalar multiplication, the (full) multiplication involves an intricate interaction between **matrix rows** and **matrix columns**. In light of this, it makes sense to first begin by defining what it means to multiply a row with a (compatible) column:

Definition 2.4 (Dot Multiplication)

Given an n-entry row matrix $A = \begin{pmatrix} A_1 & \cdots & A_n \end{pmatrix}$ and an n-entry column matrix $B = \begin{pmatrix} B_1 \\ \vdots \\ B_n \end{pmatrix}$, we define $A \bullet B$ as the *sum* obtained by adding up the products resulted from multiplying the corresponding entries of A and B. That is:

$$\begin{aligned} A \bullet B &\overset{df}{=} A_1 B_1 + \cdots + A_n B_n \\ &= \sum_{i=1}^{n} A_i B_i \end{aligned}$$

Remark: Dot Multiplication vs. Dot Product

Here, notice that while the definition of dot multiplication resembles that of **dot product** for vectors, these two operations actually represent slightly distinct concepts. For example:

- Dot multiplication operates solely on *matrices*, while dot product operates solely on *vectors*.
- Dot multiplication is not **commutative** (i.e., $A \bullet B \neq B \bullet A$), while dot product is.

And with that settled, the full definition of matrix multiplication is now in order:

Definition 2.5 (Multiplication)

Given a $m \times p$ matrix A and a $p \times n$ matrix B (i.e., a row of A has the same number of entries as a column of B), we define AB as the $m \times n$ matrix resulted by *dot-multiplying* the rows of A with the columns of B. That is:

$$(AB)_{ij} \overset{df}{=} A_{i\square} \bullet B_{\square j} \qquad \text{(for all } i \in \{1, \ldots, m\} \text{ and } j \in \{1, \ldots, n\}\text{)}$$

Caution: Matrix Multiplication is not Dot Multiplication

Matrix multiplication, which produces a *matrix*, should not be mistaken with dot multiplication, which produces a *scalar quantity*. In particular, even in the presence of an n-entry row matrix R and an n-entry column matrix C, we still have that:

$$RC = \Big(R \bullet C \Big) \neq R \bullet C$$

The following proposition expands upon the definition above — further demonstrating how a matrix product can be visualized under the Expanded Notational System:

Proposition 2.7 (Multiplication — Visual Representation)

Given m p-entry row matrices $R_1, \ldots, R_m$ and n p-entry column matrices $C_1, \ldots, C_n$, we have that:

$$\begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix} \begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix} = \begin{pmatrix} R_1 \bullet C_1 & \cdots & R_1 \bullet C_n \\ \vdots & & \vdots \\ R_m \bullet C_1 & \cdots & R_m \bullet C_n \end{pmatrix}$$

In particular, we have that:

$$R_1 \begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix} = \begin{pmatrix} R_1 \bullet C_1 & \cdots & R_1 \bullet C_n \end{pmatrix}$$

And that:

$$\begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix} C_1 = \begin{pmatrix} R_1 \bullet C_1 \\ \vdots \\ R_m \bullet C_1 \end{pmatrix}$$

Proof Immediate from the definition of matrix multiplication. ■

In fact, one can even carry out multiplications on **row/column representations** of matrices — as the proposition suggests:

Proposition 2.8 (Multiplication on Row/Column Representations)

Given a $m \times p$ matrix A and n p-entry column $C_1, \ldots, C_n$, we have that:

$$A \begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix} = \begin{bmatrix} AC_1 & \cdots & AC_n \end{bmatrix}$$

Similarly, given m p-entry row matrices $R_1, \ldots, R_m$ and a $p \times n$ matrix B, we also have that:

$$\begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix} B = \begin{bmatrix} R_1 B \\ \vdots \\ R_m B \end{bmatrix}$$

Proof To begin, notice that for both claims, the expressions on either sides of the equation are both $m \times n$ matrices. For the first claim, we can proceed by equating the **corresponding columns** of the two matrices.

More specifically, given a $j \in \{1, \dots, n\}$, we have that:

$$\begin{aligned}
\left(A\begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix}\right)_{\square j} &= \left(\begin{bmatrix} A_{1\square} \\ \vdots \\ A_{m\square} \end{bmatrix}\begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix}\right)_{\square j} \\
&= \begin{pmatrix} A_{1\square} \bullet C_1 & \cdots & A_{1\square} \bullet C_n \\ \vdots & & \vdots \\ A_{m\square} \bullet C_1 & \cdots & A_{m\square} \bullet C_n \end{pmatrix}_{\square j} \\
&= \begin{pmatrix} A_{1\square} \bullet C_j \\ \vdots \\ A_{m\square} \bullet C_j \end{pmatrix} \\
&= \begin{bmatrix} A_{1\square} \\ \vdots \\ A_{m\square} \end{bmatrix} C_j \\
&= AC_j \\
&= \begin{bmatrix} AC_1 & \cdots & AC_n \end{bmatrix}_{\square j}
\end{aligned}$$

And since j is arbitrary, the proof of the first claim is complete. Note that the second claim can be proved similarly by equating the **corresponding row** of the two matrices as well.

■

In particular, we have that:

Proposition 2.9 (Multiplication on Matrix Rows and Matrix Columns)

Given a $m \times p$ matrix A and a $p \times n$ matrix B, the following two claims hold:

(i) $(AB)_{i\square} = A_{i\square}B$ for all $i \in \{1, \dots, m\}$.

(ii) $(AB)_{\square j} = AB_{\square j}$ for all $j \in \{1, \dots, n\}$.

Proof For (i), notice that given any $i \in \{1, \ldots, m\}$, we have that:

$$\begin{aligned}(AB)_{i\square} &= \left(\begin{bmatrix} A_{1\square} \\ \vdots \\ A_{m\square} \end{bmatrix} B \right)_{i\square} \\ &= \begin{bmatrix} A_{1\square}B \\ \vdots \\ A_{m\square}B \end{bmatrix}_{i\square} \\ &= A_{i\square}B\end{aligned}$$

By breaking B down into its **column representation**, we can also prove (ii) in a very similar manner. ■

Algebraically, this means that when it comes to matrix products, a **row index** can be pushed in to the *left matrix*, while a **column index** can be pushed in to the *right matrix*.

2.4.2 Other Key Properties

In this section, we look at other key matrix properties involving the multiplication operator — through the lens of Expanded Notational System and other results established in the previous sections.

Definition 2.6 (Identity Matrix)

Given a natural number n, we define the **identity matrix** I_n (or simply I when the context is clear) as the $n \times n$ matrix such that for all $i, j \in \{1, \ldots, n\}$:

$$I_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

As expected, the identity matrix possesses the so-called **identity property**, which states that:

Proposition 2.10 (Identity Property)

Given a $n \times n$ matrix A and the $n \times n$ **identity matrix** I, the following two claims hold:

(i) $AI = A$
(ii) $IA = A$

Proof To begin, notice that for both claims, the expressions on either sides of the equation are both $n \times n$ matrices. For the first claim, we can proceed by equating the **corresponding entries** of the two matrices.

More specifically, given $i, j \in \{1, \ldots, n\}$, we have that:

$$\begin{aligned}(AI)_{ij} &= A_{i\square} \bullet I_{\square j} \\ &= \begin{pmatrix} A_{i1} & \cdots & A_{in} \end{pmatrix} \bullet \begin{pmatrix} I_{1j} \\ \vdots \\ I_{nj} \end{pmatrix} \\ &= A_{ij} \qquad \text{(why?)}\end{aligned}$$

And since i and j are arbitrary, the proof of the first claim is complete. Note that the second claim can be proved similarly by equating the **corresponding entries** of the two matrices as well. ■

So I_n is to the set of $n \times n$ matrices the same way 1 is to the set of real numbers. On a similar note, here's another property involving both **multiplication** and **scalar multiplication**:

Proposition 2.11 (Scalar Attachability/Detachability)

Given a $m \times p$ matrix A, a $p \times n$ matrix B and a number k, the following two claims hold:

(i) $(kA)B = k(AB)$
(ii) $A(kB) = k(AB)$

Proof To begin, notice that for both claims, the expressions on either sides of the equation are both $m \times n$ matrices. For the first claim, we can proceed by equating the **corresponding entries** of the two matrices.

More specifically, given an $i \in \{1, \ldots, m\}$ and a $j \in \{1, \ldots, n\}$, we have that:

$$\begin{aligned}
\big((kA)B\big)_{ij} &= (kA)_{i\square} \bullet B_{\square j} \\
&= (kA_{i\square}) \bullet B_{\square j} \\
&= \begin{pmatrix} kA_{i1} & \cdots & kA_{ip} \end{pmatrix} \bullet \begin{pmatrix} B_{1j} \\ \vdots \\ B_{pj} \end{pmatrix} \\
&= k \left(\begin{pmatrix} A_{i1} & \cdots & A_{ip} \end{pmatrix} \bullet \begin{pmatrix} B_{1j} \\ \vdots \\ B_{pj} \end{pmatrix} \right) \qquad \text{(why?)} \\
&= k(A_{i\square} \bullet B_{\square j}) \\
&= k(AB)_{ij} \\
&= \big(k(AB)\big)_{ij}
\end{aligned}$$

And since i and j are arbitrary, the proof of the first claim is complete. Note that the second claim can be proved similarly by equating the **corresponding entries** of the two matrices as well. ■

In other words, when a scalar constant occurs in the context of a matrix product, it can either be *pushed towards* A or B — or be *pulled away* from them as one wishes.

Scalar multiplication aside, the multiplication operator can also interact with the addition operator, thereby producing the so-called **distributive properties**:

Proposition 2.12 (Left/Right Distributivity)

Given a $m \times p$ matrix A and two $p \times n$ matrices B and C, we have that:

$$A(B + C) = AB + AC \qquad \textbf{(Left Distributivity)}$$

Similarly, given two $m \times p$ matrices A and B along with a $p \times n$ matrix C, we also have that:

$$(A + B)C = AC + BC \qquad \textbf{(Right Distributivity)}$$

Proof To begin, notice that for both claims, the expressions on either sides of the equation are both $m \times n$ matrices. For the first claim, we can proceed by equating the **corresponding entries** of the two matrices.

More specifically, given an $i \in \{1, \ldots, m\}$ and a $j \in \{1, \ldots, n\}$, we have that:

$$\begin{aligned}
\Big(A(B+C)\Big)_{ij} &= A_{i\square} \bullet (B+C)_{\square j} \\
&= A_{i\square} \bullet (B_{\square j} + C_{\square j}) \\
&= \begin{pmatrix} A_{i1} & \cdots & A_{ip} \end{pmatrix} \bullet \begin{pmatrix} B_{1j} + C_{1j} \\ \vdots \\ B_{pj} + C_{pj} \end{pmatrix} \\
&= \begin{pmatrix} A_{i1} & \cdots & A_{ip} \end{pmatrix} \bullet \begin{pmatrix} B_{1j} \\ \vdots \\ B_{pj} \end{pmatrix} + \begin{pmatrix} A_{i1} & \cdots & A_{ip} \end{pmatrix} \bullet \begin{pmatrix} C_{1j} \\ \vdots \\ C_{pj} \end{pmatrix} \\
&= A_{i\square} \bullet B_{\square j} + A_{i\square} \bullet C_{\square j} \\
&= (AB)_{ij} + (AC)_{ij} \\
&= (AB + AC)_{ij}
\end{aligned}$$

And since i and j are arbitrary, the proof of the first claim is complete. Note that the second claim can be proved similarly by equating the **corresponding entries** of the two matrices as well.

■

In addition to scalar multiplication and addition, the multiplication operator can also interact with the **transposition operator**, leading to yet another interesting matrix property:

Proposition 2.13 (Transposition of Matrix Product)

Given a $m \times p$ matrix A and a $p \times n$ matrix B, we have that:

$$(AB)^T = B^T A^T$$

Proof To begin, notice that the expressions on either sides of the equation are both $n \times m$ matrices. With that in mind, we can proceed with the proof by equating the **corresponding entries** of the two matrices.

More specifically, given an $i \in \{1, \dots, n\}$ and a $j \in \{1, \dots, m\}$, we have that:

$$\begin{aligned}
\left((AB)^T\right)_{ij} &= (AB)_{ji} \\
&= A_{j\square} \bullet B_{\square i} \\
&= \begin{pmatrix} A_{j1} & \cdots & A_{jp} \end{pmatrix} \bullet \begin{pmatrix} B_{1i} \\ \vdots \\ B_{pi} \end{pmatrix} \\
&= \begin{pmatrix} B_{1i} & \cdots & B_{pi} \end{pmatrix} \bullet \begin{pmatrix} A_{j1} \\ \vdots \\ A_{jp} \end{pmatrix} \\
&= (B_{\square i})^T \bullet (A_{j\square})^T \\
&= (B^T)_{i\square} \bullet (A^T)_{\square j} \\
&= (B^T A^T)_{ij}
\end{aligned}$$

And since i and j are arbitrary, the proof is now complete. ■

We now conclude the section with a fundamental property on multiplication which — as easy as it is to state — is rarely proved in introductory texts:

Proposition 2.14 (Multiplicative Associativity)

Given a $m \times p$ matrix A, a $p \times q$ matrix B and a $q \times n$ matrix C, we have that:

$$(AB)C = A(BC)$$

Proof To begin, notice that the expressions on either sides of the equation are both $m \times n$ matrices. With that in mind, we can proceed with the proof by equating the **corresponding entries** of the two matrices.

More specifically, given an $i \in \{1, \dots, m\}$ and a $j \in \{1, \dots, n\}$, exploring the *left side* of the equation yields that:

$$
\begin{aligned}
\Big((AB)C\Big)_{ij} &= (AB)_{i\square} \bullet C_{\square j} \\
&= \begin{pmatrix} (AB)_{i1} & \cdots & (AB)_{iq} \end{pmatrix} \bullet \begin{pmatrix} C_{1j} \\ \vdots \\ C_{qj} \end{pmatrix} \\
&= \begin{pmatrix} A_{i\square} \bullet B_{\square 1} & \cdots & A_{i\square} \bullet B_{\square q} \end{pmatrix} \bullet \begin{pmatrix} C_{1j} \\ \vdots \\ C_{qj} \end{pmatrix} \\
&= (A_{i\square} \bullet B_{\square 1})C_{1j} + \cdots + (A_{i\square} \bullet B_{\square q})C_{qj} \\
&= \sum_{({\color{red}\bigcirc},{\color{blue}\bigcirc})=(1,1)}^{(p,q)} A_{i{\color{red}\bigcirc}} B_{{\color{red}\bigcirc}{\color{blue}\bigcirc}} C_{{\color{blue}\bigcirc} j} \qquad \text{(why?)}
\end{aligned}
$$

On the other hand, exploring the *right side* of the equation also yields that:

$$
\begin{aligned}
\Big(A(BC)\Big)_{ij} &= A_{i\square} \bullet (BC)_{\square j} \\
&= \begin{pmatrix} A_{i1} & \cdots & A_{ip} \end{pmatrix} \bullet \begin{pmatrix} (BC)_{1j} \\ \vdots \\ (BC)_{pj} \end{pmatrix} \\
&= \begin{pmatrix} A_{i1} & \cdots & A_{ip} \end{pmatrix} \bullet \begin{pmatrix} B_{1\square} \bullet C_{\square j} \\ \vdots \\ B_{p\square} \bullet C_{\square j} \end{pmatrix} \\
&= A_{i1}(B_{1\square} \bullet C_{\square j}) + \cdots + A_{ip}(B_{p\square} \bullet C_{\square j}) \\
&= \sum_{({\color{red}\bigcirc},{\color{blue}\bigcirc})=(1,1)}^{(p,q)} A_{i{\color{red}\bigcirc}} B_{{\color{red}\bigcirc}{\color{blue}\bigcirc}} C_{{\color{blue}\bigcirc} j} \qquad \text{(why?)}
\end{aligned}
$$

And since i and j are arbitrary, the proof is now complete. ■

Incidentally, this is also one of the proofs which illustrates how the Expanded Notational System — while tedious to define and develop at first — can make a derivation much simpler and shorter in the long run.

3 Final Words and Further Exploration

As we have seen in the previous sections, the Expanded Notational System — along with the properties established upon it thus far — allows us articulate matrix-related ideas with clarity and perform **algebra** on row/column representations and matrix indices.

As a result, instead of resorting to verbose mathematical expressions or ambiguous English paragraphs, we can now prove a wide range of matrix-related propositions with great **concision** — without sacrificing **intuition** or **rigor**.

While not readily apparent, the Expanded Notational System is set up in part to discourages **2D representations of matrices** (along with the verbosity commonly associated with them). In doing so, it often frees up a great deal of irrelevant details — thereby allowing us to better focus on what matters the most.

Granted, while some of the fundamental operational properties might have appeared trivial, in the grand scheme of things, each of these properties aids in further refining the **algebraic** and **inferential strength** of the system upon which we operate.

In fact, we have also seen that once the fundamental operational properties established, almost all future propositions can be proved succinctly through repeated applications of those properties. As the old adage says, "**what started hard becomes exponentially easier**", and that seems to be the case with the Expanded Notational System.

For your curiosity, here are some of the other important matrix properties the Expanded Notational System works particularly well with:

Other Important Matrix Properties

- Given two $n \times n$ matrices A and B, $\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B)$ and $\text{Tr}(AB) = \text{Tr}(BA)$.
- Given two $n \times n$ diagonal matrices A and B, AB is precisely the diagonal matrix such that $(AB)_{ii} = A_{ii}B_{ii}$ for all $i \in \{1, \ldots, n\}$.
- Given a $n \times n$ matrix A, if E and A' are the $n \times n$ matrices resulted from applying an **elementary row operation** on I_n and A, respectively, then $EA = A'$.
- Given a $n \times n$ matrix A, $\text{Adj}(A)A = A\,\text{Adj}(A) = \text{Det}(A)I_n$.
- (**Cramer's Rule**) Given an invertible $n \times n$ matrix A and two n-entry

column matrices $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and B, then $AX = B$ if and only if $x_i = \frac{\mathrm{Det}(A_i)}{\mathrm{Det}(A)}$ for all $i \in \{1, \ldots, n\}$ (where A_i is the $n \times n$ matrix obtained by replacing the i^{th} column of A by B).

- (**Diagonalizability**) Given a $n \times n$ matrix A, if A has n eigenvectors (written as column matrices V_1, $\ldots V_n$ with $\lambda_1, \ldots, \lambda_n$ being their associated eigenvalues) such that $P \overset{df}{=} \begin{bmatrix} V_1 & \cdots & V_n \end{bmatrix}$ is invertible, then $P^{-1}AP = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$.

As you can probably see by now, becoming acquainted with the Expanded Notational System — along with the properties introduced thus far — can provide you with a robust theoretical foundation on matrices. Not only will you internalize a new set of **algebraic rules**, but you will also develop a **strong intuition** on why matrix components behave and interact with each other the way they are.

Key Ideas

Expanded Notational System
- Standard Array Representation
- Row Representation
- Column Representation
- Matrix Row (Notation)
- Matrix Column (Notation)

Matrix Equivalence
- Symbolic Approaches
- Visual Approaches

Key Properties

Properties on Row/Column Representations

$$k\begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix} = \begin{bmatrix} kR_1 \\ \vdots \\ kR_m \end{bmatrix}$$

$$k\begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix} = \begin{bmatrix} kC_1 & \cdots & kC_n \end{bmatrix}$$

$$\begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix} + \begin{bmatrix} R'_1 \\ \vdots \\ R'_m \end{bmatrix} = \begin{bmatrix} R_1 + R'_1 \\ \vdots \\ R_m + R'_m \end{bmatrix}$$

$$\begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix} + \begin{bmatrix} C'_1 & \cdots & C'_n \end{bmatrix} = \begin{bmatrix} C_1 + C'_1 & \cdots & C_n + C'_n \end{bmatrix}$$

$$\begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix}^T = \begin{bmatrix} {R_1}^T & \cdots & {R_m}^T \end{bmatrix}$$

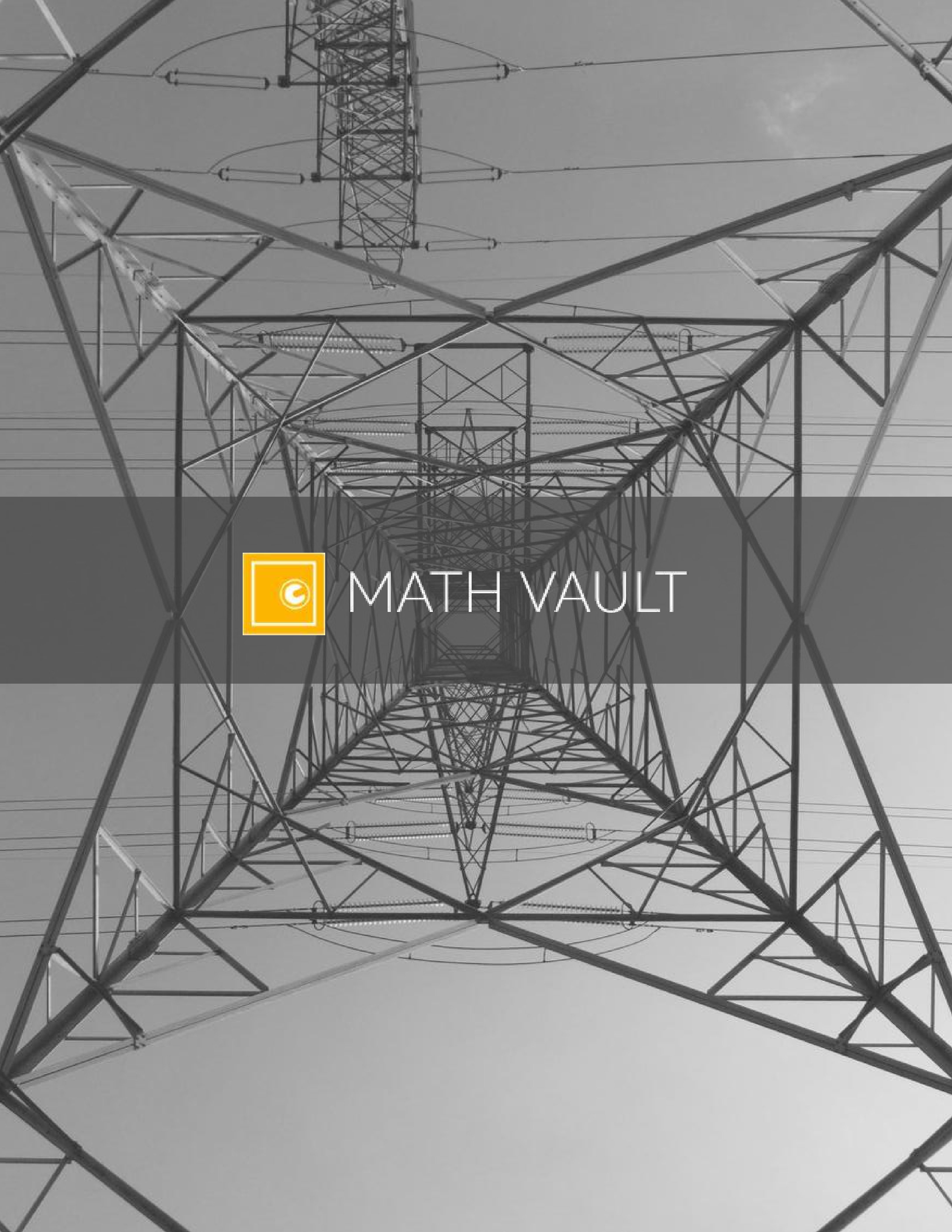
$$\begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix}^T = \begin{bmatrix} {C_1}^T \\ \vdots \\ {C_n}^T \end{bmatrix}$$

$$A\begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix} = \begin{bmatrix} AC_1 & \cdots & AC_n \end{bmatrix}$$

$$\begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix} B = \begin{bmatrix} R_1B \\ \vdots \\ R_mB \end{bmatrix}$$

Properties on Matrix Rows and Matrix Columns

- $(kA)_{i\square} = kA_{i\square}$
- $(kA)_{\square j} = kA_{\square j}$
- $(A+B)_{i\square} = A_{i\square} + B_{i\square}$
- $(A+B)_{\square j} = A_{\square j} + B_{\square j}$
- $(A^T)_{i\square} = (A_{\square i})^T$
- $(A^T)_{\square j} = (A_{j\square})^T$
- $(AB)_{i\square} = A_{i\square}B$
- $(AB)_{\square j} = AB_{\square j}$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