A Survey on Bell & Machover's *PropCal*

— Key Concepts and Structural Rules

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1 Introduction

In their 1977 seminar work *A Course in Mathematical Logic*, Bell and Machover introduce an alternate proof system for propositional logic. Referred to in the book as *Propositional Calculus* (henceforth referred to as *PropCal* for short), this linear axiomatic proof system is based solely on the 3 axiom schemes outlined below:

Axiom Scheme I	$\alpha \to (\beta \to \alpha)$	(affirming the consequent)
Axiom Scheme II	$[\alpha \to (\beta \to \gamma)] \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$	($\to$ distributivity)
Axiom Scheme III	$(\neg\alpha \to \beta) \to [(\neg\alpha \to \neg\beta) \to \alpha]$	(principle of indirect proof)

From there, Bell and Machover define a **deduction/proof** from a premise set Φ to α as a sequence of formulas $\alpha_1, \dots, \alpha_n (= \alpha)$ such that for all i between 1 and n:

1. Either α_i is in Φ, or
2. α_i is an instance of one of the aforementioned axiom schemes, or
3. α results from an application of the inference rule *Modus Ponens*.

If at least one such deduction exists, then we say that $\Phi \vdash \alpha$ (equiv., Φ proves α). Note that Φ could very well be the empty set, in which case α is called a **theorem**.

Additionally, the definition of a proof does not require that all premises in Φ be used. In fact, in the case where Φ is infinite, it is actually impossible that all premises be used.

2 Structural Rules

Despite having only 3 axiom schemes, the *PropCal* system is actually surprisingly robust in terms of deductive power. It also manages to have all the key properties any other proof system of propositional logic (that is both *sound* and *complete*) would have.

For one, the axiom schemes, when combined with the "rules of the game", give rise to a dozen of additional rules of inference — called **structural rules** — which one can readily use to further facilitate the inference process.

2.1 Definition-Based Structural Rules

In *PropCal*, there exists two key structural rules that are inherent to the definition of a deduction: *Modus Ponens* and *Cut*.

Structural Rule 1 (Modus Ponens (MP)). *Given a premise set Φ and two formulas α and β, if $\Phi \vdash \alpha \to \beta$ and $\Phi \vdash \alpha$, then $\Phi \vdash \beta$.*

Proof. Since $\Phi \vdash \alpha \to \beta$, there exists a deduction $\alpha_1, \dots, \alpha_n$ with Φ being the premise set and α_n being $\alpha \to \beta$. Similarly, since $\Phi \vdash \alpha$, there exists another deduction $\alpha_{n+1}, \dots, \alpha_{n+m}$ with Φ again being the premise set and α_{n+m} being α.

By letting α_{n+m+1} be β, one can readily see that the sequence of formulas $\alpha_1, \dots, \alpha_{n+m+1}$ constitutes a legitimate deduction from Φ to β — where the last line is justified as an application of *Modus Ponens* based on line n and line $n+m$. □

Structural Rule 2 (Cut). *If $\Phi_1 \vdash \alpha_1, \dots, \alpha_n$ and $\{\alpha_1, \dots, \alpha_n\} \cup \Phi_2 \vdash \beta$, then $\Phi_1 \cup \Phi_2 \vdash \beta$.*

Proof. By assumption, for each i between 1 and n there is a deduction from Φ_1 to α_i (i.e., a total of n deductions). Additionally, we also know that there is a deduction from $\{\alpha_1, \dots, \alpha_n\} \cup \Phi_2$ to β (say with the sequence of formulas $\beta_1, \dots, \beta_m$). With that, we proceed to construct a giant sequence of formulas as follows:

1. Combine the first n deductions together back to back, and append the last deduction $\beta_1, \dots, \beta_m$ to the very end.
2. For each β_j that is equal to some α_i, remove it from the sequence.
3. For other formulas in the last deduction citing *Modus Ponens* and referring to the aforementioned β_j, repair the citation by re-referring them to the same formulas found in the first n deductions.

By checking through the definition of deduction, one can then readily see that this giant sequence of formulas constitutes a legitimate deduction — where $\Phi_1 \cup \Phi_2$ serves as a premise set and β the conclusion. □

In practice, *Cut* says that if a first premise set proves a series of formulas, which in turn (possibly under additional premises) prove another formula, then that last formula can also be proved via the combined premise set as well. In the event where no additional premise is required, *Cut* simply boils down to the claim that the provability relation is *transitive*:

$$\text{If } \Phi \vdash \alpha_1, \ldots, \alpha_n \text{ and } \{\alpha_1, \ldots, \alpha_n\} \vdash \beta. \text{ then } \Phi \vdash \beta.$$

2.2 First Two Axiom-Based Structural Rules

At this point, all structural rules presented thus far had to be proved from first principles. However, we shall soon see that this only paves the way for the remaining rules to be proved *structurally* — effectively taking into advantage the axiomatic heavy-lifting that's being carried out a bit earlier.

As a start, we begin by noticing that the first two axiom schemes naturally give rise to their corresponding structural rules: *Affirming the Consequent* and → *Distributivity*.

Structural Rule 3 (Affirming the Consequent)**.** *Given a premise set* Φ *and two formulas* α *and* β*, if* $\Phi \vdash \alpha$*, then* $\Phi \vdash \beta \to \alpha$*.*

Proof.

$$\Phi \vdash \alpha \qquad \text{(Assumption)} \qquad (1)$$
$$\Phi \vdash \alpha \to (\beta \to \alpha) \qquad \text{(Axiom Scheme I)} \qquad (2)$$
$$\Phi \vdash \beta \to \alpha \qquad \text{(MP, (1), (2))}$$

□

Structural Rule 4 (→ Distributivity)**.** *Given a premise set* Φ *and three formulas* α*,* β *and* γ*, if* $\Phi \vdash \alpha \to (\beta \to \gamma)$*, then* $\Phi \vdash (\alpha \to \beta) \to (\alpha \to \gamma)$*.*

Proof.

$$\Phi \vdash \alpha \to (\beta \to \gamma) \qquad \text{(Assumption)} \qquad (3)$$
$$\Phi \vdash [\alpha \to (\beta \to \gamma)] \to [(\alpha \to \beta) \to (\alpha \to \gamma)] \qquad \text{(Axiom Scheme II)} \qquad (4)$$
$$\Phi \vdash (\alpha \to \beta) \to (\alpha \to \gamma) \qquad \text{(MP, (3), (4))}$$

□

2.3 The Deduction Theorem

Of course, in order for *PropCal* to be of any use, it is necessary for it to be able to prove the trivial, foundational theorems that we might require from time to time. As an illustration, here is a trivial-looking, seemingly self-explanatory structural rule whose proof is actually quite subtle and requires a bit of ingenuity.

Structural Rule 5 (Self-Implicability)**.** *Given a premise set Φ and a formula α, $\Phi \vdash \alpha \to \alpha$.*

Proof. By hacking the first axiom scheme repeatedly and invoking $\to$ *distributivity* and MP, we get that:

$$
\begin{aligned}
\Phi &\vdash \alpha \to [(\alpha \to \alpha) \to \alpha] && \text{(Axiom Scheme I)} \quad (5)\\
&\vdash [\alpha \to (\alpha \to \alpha)] \to (\alpha \to \alpha) && (\to \text{ distributivity, (5)}) \quad (6)\\
&\vdash \alpha \to (\alpha \to \alpha) && \text{(Axiom Scheme I)} \quad (7)\\
&\vdash \alpha \to \alpha && \text{(MP, (6), (7))}
\end{aligned}
$$

□

Since our formulas are built in a huge part using $\to$, it is not unreasonable to expect its corresponding proof system to reflect the meaning of $\to$ as well. In fact, the following structural rule shows our system is indeed well-suited for that purpose.

Structural Rule 6 (Deduction Theorem (DT))**.** *Given a premise set Φ and two formulas α and β, if $\Phi \cup \{\alpha\} \vdash \beta$, then $\Phi \vdash \alpha \to \beta$.*

Proof. By assumption, there is a deduction $\beta_1, \dots, \beta_n$ with premise set $\Phi \cup \{\alpha\}$ and conclusion β, so we do know — by the definition of a deduction — that $\Phi \cup \{\alpha\} \vdash \beta_i$ for each i between 1 and n.

In a similar spirit, it turns out that one can also show — by strong induction — that $\Phi \vdash \alpha \to \beta_i$ for all i between 1 and n:

Base Case: We want to show that $\Phi \vdash \alpha \to \beta_1$. Since by assumption β_1 is the first formula in a deduction, it couldn't have possibly resulted from an application of *Modus Ponens*. This would leave us with two possible scenarios: either β_1 belongs to the premise set $\Phi \cup \{\alpha\}$, or it is an instance of one of the 3 axiom schemes.

- $\beta_1 \in \Phi \cup \{\alpha\}$: If $\beta_1 \in \Phi$, then clearly $\Phi \vdash \beta_1$, which means that $\Phi \vdash \alpha \to \beta_1$ by *Affirming the Consequent*. If on the other hand we have that $\beta_1 = \alpha$, then $\Phi \vdash \alpha \to \beta_1$ would still follow by *Self-Implicability*.

- β_1 is an axiom: In this case, since clearly $\Phi \vdash \beta_1$, applying *Affirming the Consequent* again would still show that $\Phi \vdash \alpha \to \beta_1$ — thereby exhausting all the scenarios for the base case.

Inductive Case: Assume — by inductive hypothesis — that $\Phi \vdash \alpha \to \beta_1, \dots, \Phi \vdash \alpha \to \beta_m$ (where m is an arbitrary positive integer smaller than n). Our goal is to show that $\Phi \vdash \alpha \to \beta_{m+1}$ as well.

If β_{m+1} is a premise or an axiom, then the above argument in the base case could be reused to show that $\Phi \vdash \alpha \to \beta_{m+1}$. If on the other hand β_{m+1} results from an application of *Modus Ponens*, then it would mean that there exists two positive integers i and j (both less than $m+1$) such that for some formulas ϕ_1 and ϕ_2:

1. $\beta_i = \phi_1$
2. $\beta_j = \phi_1 \to \phi_2$
3. $\beta_{m+1} = \phi_2$

In which case, it would follow that:

$$\begin{aligned}
&\Phi \vdash \alpha \to \beta_i \quad (\text{i.e., } \alpha \to \phi_1) && \text{(Inductive Hypothesis)} \quad (8)\\
&\Phi \vdash \alpha \to \beta_j \quad (\text{i.e., } \alpha \to (\phi_1 \to \phi_2)) && \text{(Inductive Hypothesis)} \quad (9)\\
&\Phi \vdash (\alpha \to \phi_1) \to (\alpha \to \phi_2) && (\to \text{Distributivity, } (9)) \quad (10)\\
&\Phi \vdash (\alpha \to \phi_2) \quad (\text{i.e., } \alpha \to \beta_{m+1}) && (\text{MP, } (8), (10))
\end{aligned}$$

So $\Phi \vdash \alpha \to \beta_{m+1}$, thereby settling the inductive case. Hence we must have that, $\Phi \vdash \alpha \to \beta_i$ for all i between 1 and n. In particular, $\Phi \vdash \alpha \to \beta_n$ (i.e., $\Phi \vdash \alpha \to \beta$). □

In a nutshell, DT formalizes our intuitive understanding about an implication, and represents the first structural rule that legitimizes the so-called *discharge of assumptions*. As such, it plays an integral role in establishing proofs where making temporary assumptions is a necessary evil.

2.4 Inconsistency-Based Structural Rules

Since some proof techniques rely heavily on the presence of contradiction, it is not unusual for a proof system to have some sort of proof-by-contradiction rules embedded into it.

In *PropCal*, if a premise set Φ is such that $\Phi \vdash \alpha, \neg\alpha$ for some formula α, then Φ is called an **inconsistent set** (equiv., $\Phi \vdash$). As one would expect, there are quite a few structural rules whose working depends on the presence of inconsistency as well.

Structural Rule 7 (Principle of Indirect Proof (PIP))**.** *Given a premise set Φ and a formula α, if $\Phi \cup \{\neg\alpha\} \vdash$, then $\Phi \vdash \alpha$.*

Proof.

$$
\begin{aligned}
\Phi \cup \{\neg\alpha\} &\vdash \beta, \neg\beta \quad \text{(for some formula } \beta) && \text{(definition of inconsistency)} && (11)\\
\Phi &\vdash \neg\alpha \to \beta, \neg\alpha \to \neg\beta && \text{(DT, (11))} && (12)\\
\Phi &\vdash (\neg\alpha \to \beta) \to [(\neg\alpha \to \neg\beta) \to \alpha] && \text{(Axiom Scheme III)} && (13)\\
\Phi &\vdash (\neg\alpha \to \neg\beta) \to \alpha && \text{(MP, (12), (13))} && (14)\\
\Phi &\vdash \alpha && \text{(MP, (12),(14))} &&
\end{aligned}
$$

□

In essence, PIP is really just the third axiom scheme in disguise. However, unlike the structural rules corresponding to the first two axiom schemes, proving PIP actually requires the heavy machinery that DT provides.

With the three axiom schemes readily converted into structural rules, we can now refrain from invoking the schemes altogether. And with just a bit of work, the other inconsistency-based structural rules can be derived one after another as well.

Structural Rule 8 (Inconsistency Effect (IE))**.** *Given a premise set Φ and a formula α, if $\Phi \vdash$, then $\Phi \vdash \alpha$.*

Proof.

$$
\begin{aligned}
\Phi &\vdash \beta, \neg\beta \quad \text{(for some formula } \beta) && \text{(definition of inconsistency)} &&\\
\Phi \cup \{\neg\alpha\} &\vdash \beta, \neg\beta && \text{(definition of deduction)} && (15)\\
\Phi &\vdash \alpha && \text{(PIP, (15))} &&
\end{aligned}
$$

□

Now, since a formula with double negation is tautologically equivalent to the same formula with the double negation eliminated, one would think that a robust proof system such as *PropCal* should to be able to "remove" the double negation as one navigates through the deduction process — a hunch that is well-confirmed by the following structural rule.

Structural Rule 9 (Double Negation Elimination)**.** *Given a premise set Φ and a formula α, if $\Phi \vdash \neg\neg\alpha$, then $\Phi \vdash \alpha$.*

Proof.

$$\Phi \cup \{\neg\alpha\} \vdash \neg\alpha, \neg\neg\alpha \qquad \text{(definition of deduction)} \tag{16}$$
$$\Phi \vdash \alpha \qquad \text{(PIP, (16))}$$

□

At this point, we have a proof-by-contradiction rule that allows for *removing* the negation sign in the inference process (i.e., PIP), but is there a similar rule out there that allows for *adding* a negation sign instead? As it turns out, a structural rule corresponding to the counterpart of PIP does indeed exist — thereby showing that proof by contradiction in *PropCal* does indeed work in both ways.

Structural Rule 10 (Reductio)**.** *Given a premise set* Φ *and a formula* α*, if* $\Phi \cup \{\alpha\} \vdash$*, then* $\Phi \vdash \neg\alpha$*.*

Proof.

$$\Phi \cup \{\neg\neg\alpha\} \vdash \neg\neg\alpha \qquad \text{(definition of deduction)} \tag{17}$$
$$\Phi \cup \{\neg\neg\alpha\} \vdash \alpha \qquad \text{(Double Negation Elimination, (17))} \tag{18}$$
$$\Phi \cup \{\alpha\} \vdash \beta, \neg\beta \quad \text{(for some formula } \beta\text{)} \qquad \text{(definition of inconsistency)} \tag{19}$$
$$\Phi \cup \{\neg\neg\alpha\} \vdash \beta, \neg\beta \qquad \text{(Cut, (18), (19))} \tag{20}$$
$$\Phi \vdash \neg\alpha \qquad \text{(PIP, (20))}$$

□

2.5 Two Higher Structural Rules

In Machover's *Set Theory, Logic and their Limitations*, the inference process known as **Denying the Consequent** is embedded into the fourth axiom scheme. In the current version of *PropCal*, however, validating it as a structural rule requires that we establish IE and DT beforehand.

Structural Rule 11 (Denying the Antecedent)**.** *Given a premise set* Φ *and two formulas* α *and* β*, if* $\Phi \vdash \neg\alpha$*, then* $\Phi \vdash \alpha \to \beta$*.*

Proof.

$$\Phi \cup \{\alpha\} \vdash \alpha \qquad \text{(definition of deduction)} \tag{21}$$
$$\Phi \cup \{\alpha\} \vdash \neg\alpha \qquad \text{(assumption / definition of deduction)} \tag{22}$$
$$\Phi \cup \{\alpha\} \vdash \beta \qquad \text{(IE,(21), (22))} \tag{23}$$
$$\Phi \vdash \alpha \to \beta \qquad \text{(DT, (23))}$$

□

As one would expect, *Denying the Antecedent* and *Affirming the Consequent* go hand in hand, with both expressing some intuitive facts about how the truth of an implication is related to the truth of its constituents: whereas *Denying the Antecedent* asserts that "if a statement is false, it implies anything", *Affirming the Consequent* articulates the idea that "if a statement is true, then anything implies it".

As a recap, all the formulas in *PropCal* (hence all the axioms as well) are constructed using $\neg$ and $\rightarrow$. Accordingly, it tends to accomodate proof techniques such as *direct proof* (e.g., MP, Cut, DT) and *indirect proof* (e.g., PIP, Reductio) fairly well. On the other hand, it's not all clear how such a system can accommodate other proof techniques such as *proof by cases*, when the language from which the formulas are constructed does not even contain the disjunction connective $\vee$.

As luck would have it, it turns out that *PropCal* is well capable of formalizing the inference process of proof by cases — and sometimes even without resorting to any formula used to simulate disjunction. To illustrate, here is a structural rule which corresponds to a special form of proof by cases — the one that conditions on the truth and the falsity of a *single* formula.

Structural Rule 12 (Proof by Cases)**.** *Given a premise set Φ and two formulas α and β, if $\Phi \cup \{\alpha\} \vdash \beta$ and $\Phi \cup \{\neg\alpha\} \vdash \beta$, then $\Phi \vdash \beta$.*

Proof.

$$\Phi \cup \{\alpha\} \cup \{\neg\beta\} \vdash \beta, \neg\beta \qquad \text{(assumption / definition of deduction)} \tag{24}$$

$$\Phi \cup \{\neg\beta\} \vdash \neg\alpha \qquad \text{(Reductio, (24))} \tag{25}$$

$$\Phi \cup \{\neg\alpha\} \cup \{\neg\beta\} \vdash \beta, \neg\beta \qquad \text{(assumption / definition of deduction)} \tag{26}$$

$$\Phi \cup \{\neg\beta\} \vdash \alpha \qquad \text{(PIP, (26))} \tag{27}$$

$$\Phi \vdash \beta \qquad \text{(PIP, (25), (27))}$$

□

3 Recapitulation

Putting everything together, we now present all the axiom schemes and structural rules one can readily employ in *PropCal*:

Name (Axiom Scheme)	Description
AXIOM SCHEME I (AFFIRMING THE CONSEQUENT)	$\alpha \to (\beta \to \alpha)$
AXIOM SCHEME II ($\to$ DISTRIBUTIVITY)	$[\alpha \to (\beta \to \gamma)] \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$
AXIOM SCHEME III (PRINCIPLE OF INDIRECT PROOF)	$(\neg\alpha \to \beta) \to [(\neg\alpha \to \neg\beta) \to \alpha]$

Name (Structural Rule)	Description
MODUS PONENS (MP)	If $\Phi \vdash \alpha \to \beta$ and $\Phi \vdash \alpha$, then $\Phi \vdash \beta$.
CUT	If $\Phi_1 \vdash \alpha_1, \dots, \alpha_n$ and $\{\alpha_1, \dots, \alpha_n\} \cup \Phi_2 \vdash \beta$, then $\Phi_1 \cup \Phi_2 \vdash \beta$.
AFFIRMING THE CONSEQUENT	If $\Phi \vdash \alpha$, then $\Phi \vdash \beta \to \alpha$.
$\to$ DISTRIBUTIVITY	If $\Phi \vdash \alpha \to (\beta \to \gamma)$, then $\Phi \vdash (\alpha \to \beta) \to (\alpha \to \gamma)$.
SELF-IMPLICABILITY	$\Phi \vdash \alpha \to \alpha$
DEDUCTION THEOREM (DT)	If $\Phi \cup \{\alpha\} \vdash \beta$, then $\Phi \vdash \alpha \to \beta$.
PRINCIPLE OF INDIRECT PROOF (PIP)	If $\Phi \cup \{\neg\alpha\} \vdash$, then $\Phi \vdash \alpha$.
INCONSISTENCY EFFECT (IE)	If $\Phi \vdash$, then $\Phi \vdash \alpha$.
DOUBLE NEGATION ELIMINATION	If $\Phi \vdash \neg\neg\alpha$, then $\Phi \vdash \alpha$.
REDUCTIO	If $\Phi \cup \{\alpha\} \vdash$, then $\Phi \vdash \neg\alpha$.
DENYING THE ANTECEDENT	If $\Phi \vdash \neg\alpha$, then $\Phi \vdash \alpha \to \beta$.
PROOF BY CASES	If $\Phi \cup \{\alpha\} \vdash \beta$ and $\Phi \cup \{\neg\alpha\} \vdash \beta$, then $\Phi \vdash \beta$.